

# NAG Fortran Library Routine Document

## F08ZSF (ZGGQRF)

**Note:** before using this routine, please read the Users' Note for your implementation to check the interpretation of *bold italicised* terms and other implementation-dependent details.

### 1 Purpose

F08ZSF (ZGGQRF) computes a generalized  $QR$  factorization of a complex matrix pair  $(A, B)$ , where  $A$  is an  $n$  by  $m$  matrix and  $B$  is an  $n$  by  $p$  matrix.

### 2 Specification

```

SUBROUTINE F08ZSF (N, M, P, A, LDA, TAUA, B, LDB, TAUB, WORK, LWORK,
1                INFO)
    INTEGER          N, M, P, LDA, LDB, LWORK, INFO
    complex*16     A(LDA,*), TAUA(*), B(LDB,*), TAUB(*), WORK(*)

```

The routine may be called by its LAPACK name *zggqrf*.

### 3 Description

F08ZSF (ZGGQRF) forms the generalized  $QR$  factorization of an  $n$  by  $m$  matrix  $A$  and an  $n$  by  $p$  matrix  $B$

$$A = QR, \quad B = QTZ,$$

where  $Q$  is an  $n$  by  $n$  unitary matrix,  $Z$  is a  $p$  by  $p$  unitary matrix and  $R$  and  $T$  are of the form

$$R = \begin{cases} \begin{matrix} m \\ n-m \end{matrix} \begin{pmatrix} R_{11} \\ 0 \end{pmatrix}, & \text{if } n \geq m; \\ \begin{matrix} n & m-n \\ n \end{matrix} \begin{pmatrix} R_{11} & R_{12} \end{pmatrix}, & \text{if } n < m, \end{cases}$$

with  $R_{11}$  upper triangular,

$$T = \begin{cases} \begin{matrix} p-n & n \\ n \end{matrix} \begin{pmatrix} 0 & T_{12} \end{pmatrix}, & \text{if } n \leq p, \\ \begin{matrix} n-p & p \\ p \end{matrix} \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}, & \text{if } n > p, \end{cases}$$

with  $T_{12}$  or  $T_{21}$  upper triangular.

In particular, if  $B$  is square and nonsingular, the generalized  $QR$  factorization of  $A$  and  $B$  implicitly gives the  $QR$  factorization of  $B^{-1}A$  as

$$B^{-1}A = Z^H(T^{-1}R).$$

### 4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D (1999) *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia URL: <http://www.netlib.org/lapack/lug>

Anderson E, Bai Z and Dongarra J (1992) Generalized  $QR$  factorization and its applications *Linear Algebra Appl. (Volume 162–164)* 243–271

Hammarling S (1987) The numerical solution of the general Gauss-Markov linear model *Mathematics in Signal Processing* (ed T S Durrani, J B Abbiss, J E Hudson, R N Madan, J G McWhirter, and T A Moore) 441–456 Oxford University Press

Paige C C (1990) Some aspects of generalized *QR* factorizations *Mathematics in Signal Processing* (ed M G Cox and S Hammarling) 73–91 Oxford University Press

## 5 Parameters

- 1: N – INTEGER *Input*  
*On entry:*  $n$ , the number of rows of the matrices  $A$  and  $B$ .  
*Constraint:*  $N \geq 0$ .
- 2: M – INTEGER *Input*  
*On entry:*  $m$ , the number of columns of the matrix  $A$ .  
*Constraint:*  $M \geq 0$ .
- 3: P – INTEGER *Input*  
*On entry:*  $p$ , the number of columns of the matrix  $B$ .  
*Constraint:*  $P \geq 0$ .
- 4: A(LDA,\*) – **complex\*16** array *Input/Output*  
**Note:** the second dimension of the array  $A$  must be at least  $\max(1, M)$ .  
*On entry:* the  $n$  by  $m$  matrix  $A$ .  
*On exit:* the elements on and above the diagonal of the array contain the  $\min(n, m)$  by  $m$  upper trapezoidal matrix  $R$  ( $R$  is upper triangular if  $n \geq m$ ); the elements below the diagonal, with the array TAUA, represent the unitary matrix  $Q$  as a product of  $\min(n, m)$  elementary reflectors (see Section 3.3.6 in the F08 Chapter Introduction).
- 5: LDA – INTEGER *Input*  
*On entry:* the first dimension of the array  $A$  as declared in the (sub)program from which F08ZSF (ZGGQRF) is called.  
*Constraint:*  $LDA \geq \max(1, N)$ .
- 6: TAUA(\*) – **complex\*16** array *Output*  
**Note:** the dimension of the array TAUA must be at least  $\max(1, \min(N, M))$ .  
*On exit:* the scalar factors of the elementary reflectors which represent the unitary matrix  $Q$ .
- 7: B(LDB,\*) – **complex\*16** array *Input/Output*  
**Note:** the second dimension of the array  $B$  must be at least  $\max(1, P)$ .  
*On entry:* the  $n$  by  $p$  matrix  $B$ .  
*On exit:* if  $n \leq p$ , the upper triangle of the subarray  $B(1 : n, p - n + 1 : p)$  contains the  $n$  by  $n$  upper triangular matrix  $T_{12}$ .  
If  $n > p$ , the elements on and above the  $(n - p)$ th subdiagonal contain the  $n$  by  $p$  upper trapezoidal matrix  $T$ ; the remaining elements, with the array TAUB, represent the unitary matrix  $Z$  as a product of elementary reflectors (see Section 3.3.6 in the F08 Chapter Introduction).

- 8: LDB – INTEGER *Input*  
*On entry:* the first dimension of the array B as declared in the (sub)program from which F08ZSF (ZGGQRF) is called.  
*Constraint:*  $LDB \geq \max(1, N)$ .
- 9: TAUB(\*) – **complex\*16** array *Output*  
**Note:** the dimension of the array TAUB must be at least  $\max(1, \min(N, P))$ .  
*On exit:* the scalar factors of the elementary reflectors which represent the unitary matrix Z.
- 10: WORK(\*) – **complex\*16** array *Workspace*  
**Note:** the dimension of the array WORK must be at least  $\max(1, LWORK)$ .  
*On exit:* if INFO = 0, the real part of WORK(1) contains the minimum value of LWORK required for optimal performance.
- 11: LWORK – INTEGER *Input*  
*On entry:* the dimension of the array WORK as declared in the (sub)program from which F08ZSF (ZGGQRF) is called.  
 If LWORK = -1, a workspace query is assumed; the routine only calculates the optimal size of the WORK array, returns this value as the first entry of the WORK array, and no error message related to LWORK is issued.  
*Suggested value:* for optimal performance,  $LWORK \geq \max(N, M, P) \times \max(nb1, nb2, nb3)$ , where *nb1* is the optimal **block size** for the QR factorization of an *n* by *m* matrix, *nb2* is the optimal **block size** for the RQ factorization of an *n* by *p* matrix, and *nb3* is the optimal **block size** for a call of F08AUF (ZUNMQR).  
*Constraint:*  $LWORK \geq \max(1, N, M, P)$  or LWORK = -1.
- 12: INFO – INTEGER *Output*  
*On exit:* INFO = 0 unless the routine detects an error (see Section 6).

## 6 Error Indicators and Warnings

Errors or warnings detected by the routine:

INFO < 0

If INFO = -*i*, the *i*th parameter had an illegal value. An explanatory message is output, and execution of the program is terminated.

## 7 Accuracy

The computed generalized QR factorization is the exact factorization for nearby matrices  $(A + E)$  and  $(B + F)$ , where

$$\|E\|_2 = O\epsilon\|A\|_2 \quad \text{and} \quad \|F\|_2 = O\epsilon\|B\|_2,$$

and  $\epsilon$  is the **machine precision**.

## 8 Further Comments

The unitary matrices  $Q$  and  $Z$  may be formed explicitly by calls to F08ATF (ZUNGQR) and F08CWF (ZUNGRQ) respectively. F08AUF (ZUNMQR) may be used to multiply  $Q$  by another matrix and F08CXF (ZUNMRQ) may be used to multiply  $Z$  by another matrix.

The real analogue of this routine is F08ZEF (DGGQRF).

## 9 Example

This example solves the general Gauss-Markov linear model problem

$$\min_x \|y\|_2 \quad \text{subject to} \quad d = Ax + By$$

where

$$A = \begin{pmatrix} 0.96 - 0.81i & -0.03 + 0.96i & -0.91 + 2.06i \\ -0.98 + 1.98i & -1.20 + 0.19i & -0.66 + 0.42i \\ 0.62 - 0.46i & 1.01 + 0.02i & 0.63 - 0.17i \\ 1.08 - 0.28i & 0.20 - 0.12i & -0.07 + 1.23i \end{pmatrix},$$

$$B = \begin{pmatrix} 0.5 - 1.0i & 0 & 0 & 0 \\ 0 & 1.0 - 2.0i & 0 & 0 \\ 0 & 0 & 2.0 - 3.0i & 0 \\ 0 & 0 & 0 & 5.0 - 4.0i \end{pmatrix} \quad \text{and}$$

$$d = \begin{pmatrix} 6.00 - 0.40i \\ -5.27 + 0.90i \\ 2.72 - 2.13i \\ -1.30 - 2.80i \end{pmatrix}.$$

The solution is obtained by first computing a generalized  $QR$  factorization of the matrix pair  $(A, B)$ . The example illustrates the general solution process, although the above data corresponds to a simple weighted least squares problem.

Note that the block size (NB) of 64 assumed in this example is not realistic for such a small problem, but should be suitable for large problems.

### 9.1 Program Text

```
*      F08ZSF Example Program Text
*      Mark 21 Release. NAG Copyright 2004.
*      .. Parameters ..
INTEGER          NIN, NOUT
PARAMETER       (NIN=5,NOUT=6)
INTEGER          MMAX, NB, NMAX, PMAX
PARAMETER       (MMAX=10,NB=64,NMAX=10,PMAX=10)
INTEGER          LDA, LDB, LWORK
PARAMETER       (LDA=MMAX,LDB=MMAX,LWORK=NB*(MMAX+PMAX))
COMPLEX *16     ONE, ZERO
PARAMETER       (ONE=(1.0D0,0.0D0),ZERO=(0.0D0,0.0D0))
*      .. Local Scalars ..
DOUBLE PRECISION RNORM
INTEGER          I, INFO, J, M, N, P
*      .. Local Arrays ..
COMPLEX *16     A(LDA,MMAX), B(LDB,PMAX), D(NMAX), TAUA(MMAX),
+              TAUB(MMAX+PMAX), WORK(LWORK), Y(PMAX)
*      .. External Functions ..
DOUBLE PRECISION DZNRM2
EXTERNAL        DZNRM2
*      .. External Subroutines ..
EXTERNAL        F06HBF, ZCOPY, ZGEMV, ZGGQRF, ZTRTRS, ZUNMQR,
+              ZUNMRQ
*      .. Intrinsic Functions ..
INTRINSIC       MAX, MIN
*      .. Executable Statements ..
WRITE (NOUT,*) 'F08ZSF Example Program Results'
WRITE (NOUT,*)
*      Skip heading in data file
READ (NIN,*)
READ (NIN,*) N, M, P
IF (M.LE.MMAX .AND. N.LE.NMAX .AND. P.LE.PMAX .AND. M.LE.N .AND.
+   N.LE.(M+P)) THEN
```

```

*
*   Read A, B and D from data file
*
*   READ (NIN,*) ((A(I,J),J=1,M),I=1,N)
*   READ (NIN,*) ((B(I,J),J=1,P),I=1,N)
*   READ (NIN,*) (D(I),I=1,N)
*
*   Compute the generalized QR factorization of (A,B) as
*   A = Q*(R),   B = Q*(T11 T12)*Z
*               (0)       ( 0  T22)
*
*   CALL ZGGQRF(N,M,P,A,LDA,TAUA,B,LDB,TAUB,WORK,LWORK,INFO)
*
*   Compute c = (c1) = (Q**H)*d, storing the result in D
*               (c2)
*
*   CALL ZUNMRQ('Left','Conjugate transpose',N,1,M,A,LDA,TAUA,D,N,
+             WORK,LWORK,INFO)
*
*   Putting Z*y = w = (w1), set w1 = 0, storing the result in Y1
*                   (w2)
*
*   CALL F06HBF(M+P-N,ZERO,Y,1)
*
*   IF (N.GT.M) THEN
*
*       Copy c2 into Y2
*
*       CALL ZCOPY(N-M,D(M+1),1,Y(M+P-N+1),1)
*
*       Solve T22*w2 = c2 for w2, storing result in Y2
*
*       CALL ZTRTRS('Upper','No transpose','Non-unit',N-M,1,
+             B(M+1,M+P-N+1),LDB,Y(M+P-N+1),N-M,INFO)
*
*       IF (INFO.GT.0) THEN
*           WRITE (NOUT,*)
+             'The upper triangular factor, T22, of B is singular, '
*           WRITE (NOUT,*)
+             'the least squares solution could not be computed'
*           GO TO 20
*       END IF
*
*       Compute estimate of the square root of the residual sum of
*       squares norm(y) = norm(w2)
*
*       RNORM = DZNRM2(N-M,Y(M+P-N+1),1)
*
*       Form c1 - T12*w2 in D
*
*       CALL ZGEMV('No transpose',M,N-M,-ONE,B(1,M+P-N+1),LDB,
+             Y(M+P-N+1),1,ONE,D,1)
*       END IF
*
*       Solve R*x = c1 - T12*w2 for x
*
*       CALL ZTRTRS('Upper','No transpose','Non-unit',M,1,A,LDA,D,M,
+             INFO)
*
*       IF (INFO.GT.0) THEN
*           WRITE (NOUT,*)
+             'The upper triangular factor, R, of A is singular, '
*           WRITE (NOUT,*)
+             'the least squares solution could not be computed'
*           GO TO 20
*       END IF
*
*       Compute y = (Z**H)*w
*
*       CALL ZUNMRQ('Left','Conjugate transpose',P,1,MIN(N,P),
+             B(MAX(1,N-P+1),1),LDB,TAUB,Y,P,WORK,LWORK,INFO)

```

```

*
*   Print least squares solution x
*
*   WRITE (NOUT,*) 'Generalized least squares solution'
*   WRITE (NOUT,99999) (D(I),I=1,M)
*
*   Print residual vector y
*
*   WRITE (NOUT,*)
*   WRITE (NOUT,*) 'Residual vector'
*   WRITE (NOUT,99998) (Y(I),I=1,P)
*
*   Print estimate of the square root of the residual sum of
*   squares
*
*   WRITE (NOUT,*)
*   WRITE (NOUT,*) 'Square root of the residual sum of squares'
*   WRITE (NOUT,99997) RNORM
*   ELSE
*   WRITE (NOUT,*)
*   +   'One or more of MMAX, NMAX or PMAX is too small, ',
*   +   'and/or M.LT.N or N.GT.(M+P)'
*   END IF
20 CONTINUE
STOP
*
99999 FORMAT (3(' (',F9.4,',',F9.4,')',:))
99998 FORMAT (3(' (',1P,E9.2,',',1P,E9.2,')',:))
99997 FORMAT (1X,1P,E10.2)
END

```

## 9.2 Program Data

F08ZSF Example Program Data

```

4           3           4                               :Values of M, N and P

( 0.96,-0.81) (-0.03, 0.96) (-0.91, 2.06)
(-0.98, 1.98) (-1.20, 0.19) (-0.66, 0.42)
( 0.62,-0.46) ( 1.01, 0.02) ( 0.63,-0.17)
( 1.08,-0.28) ( 0.20,-0.12) (-0.07, 1.23)                               :End of matrix A

( 0.50,-1.00) ( 0.00, 0.00) ( 0.00, 0.00) ( 0.00, 0.00)
( 0.00, 0.00) ( 1.00,-2.00) ( 0.00, 0.00) ( 0.00, 0.00)
( 0.00, 0.00) ( 0.00, 0.00) ( 2.00,-3.00) ( 0.00, 0.00)
( 0.00, 0.00) ( 0.00, 0.00) ( 0.00, 0.00) ( 5.00,-4.00) :End of matrix B

( 6.00,-0.40)
(-5.27, 0.90)
( 2.72,-2.13)
(-1.30,-2.80)                               :End of vector d

```

## 9.3 Program Results

F08ZSF Example Program Results

Generalized least squares solution

```
( -0.9846, 1.9950) ( 3.9929, -4.9748) ( -3.0026, 0.9994)
```

Residual vector

```
( 1.26E-04,-4.66E-04) ( 1.11E-03,-8.61E-04) ( 3.84E-03,-1.82E-03)
( 2.03E-03, 3.02E-03)
```

Square root of the residual sum of squares

```
5.79E-03
```